Seminar

Artificial Intelligence for Games (SS19)

Game Theory - Non-cooperative games:

A basic introduction

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Contents

1 Introduction

1.1 Motivation

The first known reference of something that resembles modern game theory was a letter written in 1713 by British diplomat James Waldegrave (Bellhouse, 2007). Thoughts about how to play games optimally have been around since long before that. The link between game theory and economic theory, where game theory has been applied to a great extent ever since, was established in 1944 by von Neumann and Morgenstern in their publication *The Theory of Games and Economic Behaviour.*(Screpanti et al., 2005)

Despite its age, game theory is as relevant as it has ever been. It can be used to tackle personal problems, like buying a used car (Duronio, 2012). Furthermore, game theory can also be used to think about broader taks, for example how to tackle environmental issues (Ray and Yo Dd, 2001).

In essence, a lot of real world scenarios can be modeled, analyzed and finally solved using concepts from game theory. Thus, it is critical to understand this research field.

Additionally, game theory introduces mathematically interesting concepts and ideas, that otherwise would probably not exist and are compelling to reflect upon.

1.2 Overview

This report is divided up into two main chapter, Concepts and Experiments, embedded in the two enclosing chapters: Introduction and Conclusion.

Following the Introduction (chapter 1), the second chapter, Concepts, lays the theoretical groundwork. In section 2.1, fundamental ideas of game theory are explained. Next, the specifics of normal-form games are defined in section 2.2. Lastly in this chapter, we examine a variety of solution concepts for normal-form games.

Consequently, in chapter 3 (Experiments) we take a look at five games that were played by the audience members of the talk. There, we examine the players choices and will analyze the games using tools presented in the previous section 2.3.

Lastly, in chapter 4 we summarize the content of this report and give a brief outlook of what topics would be the next to look at in game theory.

2 Concepts

2.1 Fundamentals

Strategic games describe the basic concept of a **game** in game theory. A game consists of **players** playing the game by making decisions and taking actions. These actions are also called **strategies**. For this model to be considered a game the actions of the players affect the other players, as well as the other players actions affect themselves. Each player has **preferences** about the strategies being played in a given game. A set of played strategies is called the **action profile**. (Osborne, 2003)

One example for a strategic game would be Rock, Paper, Scissor. In this game we have two players: A and B. Each player can play the strategies: Rock, Paper or Scissor. The preferences of each player are not only determined by their own action, but also by the action of the opponent. For example, when Player A chooses Rock his preference for the opponents strategies would be: Paper $<$ Rock $<$ Scissor. If player B plays Paper, then player A loses the game. If player B choses to play Rock, then the game ends in a draw. Finally, if player B plays Scissor, player A would win the game.

Complete information games have four components that need to be present. Every player who participates in the game knows **all possible actions** of all the other players. Furthermore, every player knows **all possible outcomes** of the game. Additionally, each player knows how the combinations of **actions affects the outcome**. Lastly, every player knows the **preferences** of every player. (Tadelis, 2013)

Descriptive and normative decision theory are to different approaches of looking at a strategic game. Descriptive decision theory determines the actions real humans take in a strategic game by **conducting experiments**. On the other hand, normative decision theory looks at games from a **model-based point of view**. The players are modeled as a rational and utility-maximizing agents. Using these assumptions as a basis the games can be analyzed mathematically. (Hansson, 1994)

2.2 Normal-form games

One type of strategic games with complete information is the so called normal form game. In a normal-form game we have the same features as any strategic game as a basis. Furthermore, in a normal-form game the players choose their actions *once* and *simultaneously*. Once means there is no repetition of the game and it is a one-off situation. Simultaneous means that the players chose their actions independently. At the time of choosing their action, they do not know what the opponents chosen actions are. (Tadelis, 2013)

Normal-form games can be visualized using a payoff matrix. Table 2.1 visualizes a two player Rock, Paper, Scissor game. On the left and upper part of the table the players, in this case A and B are denoted. Going inwards, the strategies for each player are shown. In this case both players have the same set of strategies, namely: Rock, Paper and Scissor. The combination of the three strategies for each player yields a 3×3 matrix. For each combination of strategies the matrix denotes the utility for the players. The first number depicts the utility for player A, the second number the utility of player B.

		Player B		
		Rock	Paper	Scissor
Player A	Rock	0,0	-1,1	1,-1
	Paper	1,-1	0,0	-1,1
	Scissor	-1,1	1,-1	0,0

Table 2.1: Exemplary payoff matrix for a two player Rock, Paper, Scissor game.

2.3 Solution concepts

If we want to analyze a game from the normative point of view, we need some mathematical tools. We will only look at pure strategies to not exceed the scope of this paper. Furthermore, we only look at **non-cooperative** normal-form games, where each player only tries to maximize his or her own utility (Tadelis, 2013).

The components of a normal-form game need to be represented in a more mathematical way in order to be able to analyze a game. As defined in section 2.1, a game consists of a set of players $N = \{1, 2, \ldots, n\}$ with the set of pure strategies S_i being playable by player i. The action profile is a vector of strategies $\vec{s} = (s_1, s_2, \ldots, s_n)$.To model the preferences of the players a set of payoff functions $U = \{u_1, u_2, \ldots, u_n\}$ is defined. A payoff function for player i maps each possible combination of chosen strategies to a real value representing the utility: $u_i : S_1 \times S_2 \times \cdots \times S_n \to \mathbb{R}$. (Tadelis, 2013)

Best response is a solution concept where the goal is to identify the best reaction to a given action. Mathematically a strategy $s_i \in S_i$ is a best response for player i in regard to the opponent $-i$'s strategy $s_{-i} \in S_{-i}$ if:

$$u_i(s_i, s_{-i}) \geq u_i({s'}_i, s_{-i}) \forall {s'}_i \in S_i \implies bR_i(s_{-i}) = i \tag{2.1}$$

. (Tadelis, 2013)

To illustrate the best response solution concept we will use the prisoners' dilemma game as an example. The payoff matrix for the game is depicted in table 2.2. The players of the game are: $\{A, B\}$. The strategies they can play are: $S_A = S_B = \{saynothing, talk\}$. For example, given the action profile $\vec{s} = (saynothing, saynothing)$, the payoffs for player A and player B would be: $u_A = u_B = -1$.

		Player B	
		say nothing	talk
Player A	say nothing	-1,-1	-3,0
	talk	0,-3	-2,-2

Table 2.2: Payoff matrix for a prisoners' dilemma game.

We want to find the best response of player A to $s_B = (saynothing)$:

$$bR_A(saynothing) = ? \tag{2.2}$$

. In order to find the best response we have to check all of player A's strategies against players B's strategy *saynothing*, as defined in 2.1. As we calculated in 2.3 and the following calculations, the best response of player A to the opponents' strategy *saynothing* is *talk*.

$$u_A(saynothing, saynothing) = -1 \tag{2.3}$$

$$u_A(talk, saynothing) = 0 \tag{2.4}$$

$$u_A(saynothing, saynothing) < u_A(talk, saynothing) \\ \implies bR_A(saynothing) = talk \tag{2.5}$$

Dominated strategies are strategies that perform worse than one specific other strategy, regardless of the opponents actions. A strategy $s'_i \in S_i$ of player i is strictly dominated by his other strategy $s_i \in S_i$ if:

$$u_i(s_i, s_{-i}) > u_i(s'{}_i, s_{-i}) \forall s_{-i} \in S_{-i} \tag{2.6}$$

. (Tadelis, 2013)

From the point of view of the best response theorem: a dominated strategy is never a best response. Hence, a rationally acting player would never play a dominated strategy. This in turn means, that for an analysis of what "is best to do", they can be removed from the original game.

Let us take another look at the payoff matrix 2.2. We want to find out if there is dominated strategy for player A. We already calculated $u_A(saynothing, saynothing)$ and $u_A(talk, saynothing)$. And we determined that $bR_A(saynothing) = (talk)$. This means in effect that our candidate for a dominated strategy is *saynothing*.

$$u_A(saynothing, talk) = -3 \tag{2.7}$$

$$u_A(talk, talk) = -2 \tag{2.8}$$

$$u_A(saynothing, talk) < u_A(talk, talk) \implies bR_A(talk) = (talk) \tag{2.9}$$

$$bR_A(s_B) = talk \forall s_B \in S_B \tag{2.10}$$

As shown in 2.3, 2.7 and the subsequent calculations, the strategy *talk* is always the best response for player A. This in turn means that *saynothing* is never a best response. The strategy *saynothing* is therefore strictly dominated by *talk* for player A.

The Nash equilibrium is defined as " *[...] a profile of strategies such that each player's strategy is an optimal response to the other players' strategies.*" (Drew Fudenberg, 1991). Optimal response and best response are synonyms in the context of game theory. If all players play a best response in a strategy profile no player has a motivation to change his strategy. A strategy profile of pure strategies $\vec{s^\star} = (s^\star{}_1, s^\star{}_2, \ldots, s^\star{}_n) \in S$ is a Nash equilibrium if:

$$u_i(s^\star{}_i, s^\star{}_{-i}) \geq u_i(s'{}_i, s^\star{}_{-i}) \forall s'{}_i \in S_i \wedge \forall i \in N \tag{2.11}$$

(Tadelis, 2013).

We already calculated the best responses for player A. In order to find the equilibrium of the game, we need to do the same for player B. In the payoff matrix 2.3 we underlined the maximum payoffs for a given strategy of an opponent, i.e. marking all best responses. The only strategy profile where all players play their best response strategy is $\vec{s^\star} = (talk, talk)$. This strategy profile is the only Nash equilibrium for pure strategies for this game.

Note that $\vec{s'} = (saynothing, saynothing)$ is not a Nash equilibrium, even though both players would earn a higher utility. In $\vec{s'}$ each player has the motivation revise their strategy, from *saynothing* to *talk*, in order to increase their utility from -1 to 0.

		Player B	
		say nothing	talk
Player A	say nothing	-1,-1	-3,<u>0</u>
	talk	<u>0</u>,-3	<u>-2</u>,<u>-2</u>

Table 2.3: Payoff matrix for a prisoners' dilemma game with underlined best response payoffs.

3 Experiments

3.1 Method

During the presentation section of this seminar the listeners had the possibility to play some Normal-form games. The games were presented online[1]. Out of total of 15 audience members who started the game, 10 finished playing.

First a participant had to enter his or her name. After submitting the name, they were assigned a team (A or B) and a random identification code. Then the participants were presented with a payoff-matrix and a selection for the pure strategies. This user interface is shown in the graphic 3.1. The participants were asked to selected their preferred strategy, before the game was discussed. But there was no mechanism implemented to verify, if the selection happend before or after this discussion. The choices of a participant were saved after they had completed all games.

Figure 3.1: Screenshot of the GUI for the Even-Odd Game

3.2 Results and Analysis

In this section, we analyze the strategies the participants played and compare them to the choices a purely rationally acting player would make.

All of the following games are symetric, meaning the payoff only depends on the strategies played and not who played them. For a two-player symmetric game the following holds true: $u_i(s_1, s_2) = u_{-i}(s_2, s_1) \forall s_{1/2} \in S_{i/-i}$(Osborne, 2003). This means we can ignore whether a participant was player A or player B.

A shared fridge The story of this game is as follows: you and your roommate share a fridge. Both can take groceries from and add new groceries to the fridge.

[1] https://aifg.desomb.re/

The payoff matrix in table 3.1 shows the payoffs for all possible combinations of strategies the players can choose.

When analyzing the game we find that for each player the best response to the opponent's strategy Low is Low and for $High$ is $High$. Therefore, two Nash equilibria exist: (Low, Low) and $(High, High)$. From a normative point of view both strategies are valid choices for the players.

Next, lets look at the actual choices the participants of the course took. Seven people chose to buy high quality, while three chose to buy only low quality groceries (see chart 3.2).

We hypothesis that people who chose Low might have done so because they did not trust the opponent to play a strategy that would be better for both ($High$).

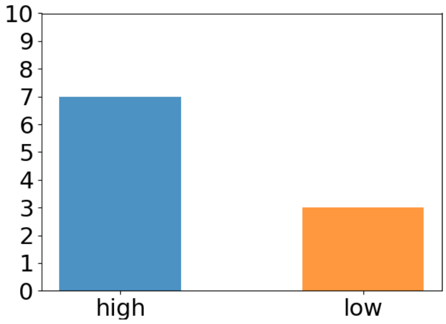

Figure 3.2: Choices of the players for the shared fridge game.

		Player B	
		Low	High
Player A	Low	$\underline{1},\underline{1}$	2,0
	High	0,2	$\underline{3},\underline{3}$

Table 3.1: Payoff matrix for the shared fridge game with underlined best response payoffs.

A dinner date The idea of the dinner date game is that the two players go out together. Before ordering their meals they decide to split the receipt evenly. The players can either choose to buy an expensive or a cheap meal. The payoffs for the game are denoted in table 3.2. For each player the best responses are: $bA_i(Expensive) = Expensive$ and $bA_i(Cheap) = Expensive$ with $i \in A, B$. This lets us determine the only Nash equilibrium of this game, namely: $(Cheap, Cheap)$. Furthermore, we can see that $Expensive$ dominates $Cheap$ strictly. Hence, choosing the strategy $Expensive$ as an action is always better than choosing $Cheap$.

Out of the ten participants of the games seven chose $Expensive$ as their strategy and three chose $Cheap$. The reason for people to choose the strategy $Cheap$ even though it is dominated can have a variety of reasons. For example, the player might have looked at the table and concluded that because $u_i(Expensive, Expensive) = u_i(Cheap, Cheap) = 2$ the choice does not matter. This conclusion is false as explained in section 2.3.

The stag hunt The stag hunt game is an often cited game in game theory. The story behind the game relates to two hunters who met the evening before the hunt.

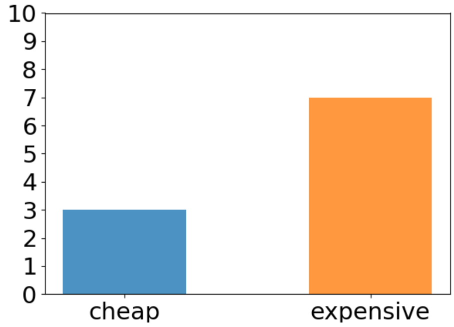

Figure 3.3: Choices of the players for the dinner date game.

		Player B	
		Expensive	Cheap
Player A	Expensive	$\underline{2},\underline{2}$	$\underline{3},0$
	Cheap	$0,\underline{3}$	2,2

Table 3.2: Payoff matrix for the dinner date game with underlined best response payoffs.

They agree upon what to do the next day, but unfortunately they drink too much and forgot what they talked about. Therefore, on the next day each hunter has to choose what he wants to do, without knowing the strategy of the other hunter.

Firstly, we determine the best responses for every strategy of the opponent. This gives us: $bA_i(Stag) = Stag$ and $bA_i(Hare) = Hare$ with $i \in A, B$. This game has two equilibria: $(Stag, Stag)$ and $(Hare, Hare)$.

When looking at the players choices we see that four people chose $Hare$ and six chose $Stag$.

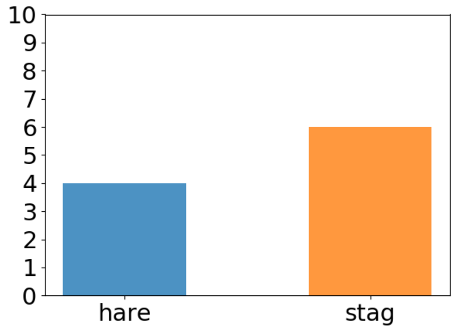

Figure 3.4: Choices of the players for the stag hunt game.

		Player B	
		Stag	Hare
Player A	Stag	$\underline{3},\underline{3}$	0,2
	Hare	2,0	$\underline{1},\underline{1}$

Table 3.3: Payoff matrix for the stag hunt game with underlined best response payoffs.

A partner project The partner project is a game about a two person group project. The partners divided up the work into two equal parts. Then they each go their separate ways until they have to hand in their joint report.

The players can either choose to $Work$ or to $Goof$. We start again by analyzing the best responses. We find that: $bR_i(Work) = Goof$ and $bR_i(Goof) = Goof$. The strategy $Work$ is therefore strictly dominated by $Goof$. Interestingly, the Nash equilibrium $(Goof, Goof)$ is worse for each individual player than $(Work, Work)$. These kinds of game are structured like the so called *prisoners' dilemma*.

A lot of the participants seem have understood the dilemma situation. Out of the ten participants eight chose to play the strategy $goof$, while only two players played $Work$.

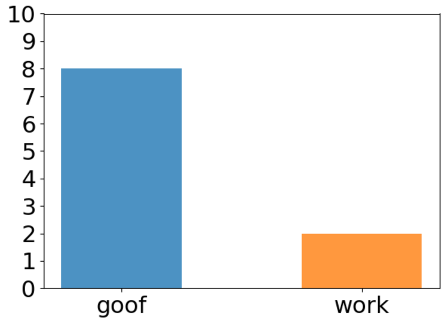

Figure 3.5: Choices of the players for the partner project game.

		Player B	
		Work	Goof
Player A	Work	2,2	0,$\underline{3}$
	Goof	$\underline{3}$,0	$\underline{1}$,$\underline{1}$

Table 3.4: Payoff matrix for the partner project game with underlined best response payoffs.

The marble game The marble game is a more complicated game. Each player has three marbles: red, blue and yellow. A player chooses his marble, puts the marble in the palm of his hand and then clenches their fist. The payoffs are determined by the combination of the colors of the marbles.

Firstly, we analyze the game from the normative point of view and begin by searching for the best responses. The best responses are: $bR_i(Red) = Blue$, $bR_i(Blue) = Blue$ and $bR_i(Yellow) = Blue$. We can see that the best response is always $Blue$. This strategy dominates all other strategies strictly.

In this case every single player played $Blue$ as their strategy, which is the equilibrium strategy.

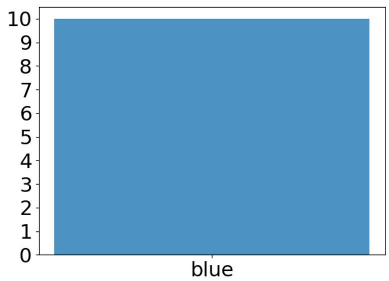

Figure 3.6: Choices of the players for the marble game.

		Player B		
		Red	Blue	Yellow
Player A	Red	2,2	0,$\underline{3}$	0,1
	Blue	$\underline{3}$,0	$\underline{2}$,$\underline{2}$	$\underline{2}$,0
	Yellow	1,0	0,$\underline{2}$	1,1

Table 3.5: Payoff matrix for the marble game with underlined best response payoffs.

4 Conclusion

4.1 Summary

In this report we firstly explained the relevance of game theory in section 1.1 by showing real world examples where it is applicable. Next, we introduced basic concepts as well as solution methods in chapter 2.

Finally, we analyzed some games from the normative point of view and compared them to the choices of real players in chapter 3. Here, we found that the choices of real people and the normative results do not always align. Furthermore, we showed that the normative solution is not always the best result for the players.

4.2 Outlook

This report was drafted as a basic introduction into game theory, because no one else chose a topic from the game theory section of the seminar. Therefore, a lot of the more complex and in-depth areas of game theory were omitted. For example, the proof of the existence of the Nash equilibrium by John Nash ((F. Nash, 1950)) was not outlined. Further subjects of game theory include:

- **mixed strategies** let players use a probability distributions over a strategy set, instead of playing deterministically (Osborne, 2003),
- in **extensive-form games** players execute their strategies in an alternating fashion (Drew Fudenberg, 1991),
- the main idea behind **repeated games** is that most situations in the real world are not one-off ones, but arise again and again (Osborne, 2003).

Furthermore, the experiments we conduct were very limited and for more representative results the conditions should be altered. Firstly, only a small number of students finished the games, namely 10. Secondly, the motivation to take the games seriously might have been limited, as the prize was only a box of chocolates.

Part I

Appendix

A Lists

A.1 List of Figures

A.2 List of Tables

B Bibliography

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